Guide to Uncertainty of Measurement

Ref.[5]

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September 28, 2016

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Outline

- Measurement
- Uncertainty of measurement
- Basic statistics
- Where do errors and uncertainties come from?
- The general kinds of uncertainty
- How to calculate uncertainty of measurement
- Making an uncertainty calculation
- Example
- Others

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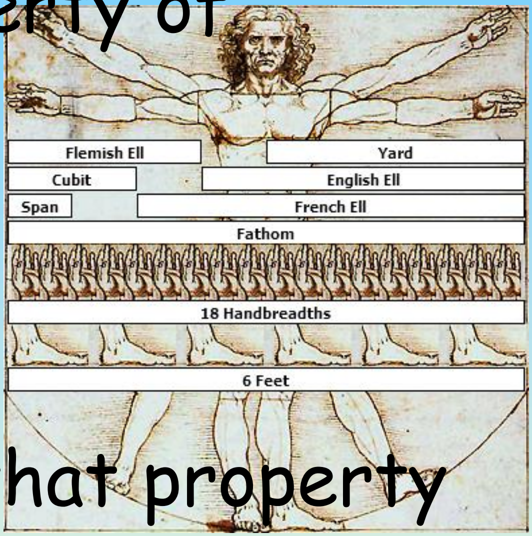
What is a measurement?



- A measurement tell us about a property of something
  - how heavy an object is
  - how the accuracy of your clock is
  - how long it is
- A measurement gives **a number** to that property
- Measurements are always made using an instrument of some kind
  - Rulers, stopwatches, weighing scales, thermometers
- The result of a measurement is normally in two parts: (a) **a number** (b) **a unit of measurement**
  - How long it is? … **2 meters**

What is not a measurement?

- There are some processes that might seem to be measurements, but are not…
  - comparing two pieces of string to see which is longer is not really a measurement

  - counting

  - tests (yes/no , pass/fail results)

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What is uncertainty of measurement and how to expressing it?

- The uncertainty of a measurement tells us something about its quality
- **Uncertainty of measurement** is the doubt that exists about the result of any measurement
- Since there is always a margin of doubt about any measurement
  - How big is the margin?
  - How bad is the doubt?
- Two numbers are really needed in order to quantify an uncertainty
  - The width of the margin **(Interval)**
  - **Confidence level**

20 cm ±1 cm, at a level of confidence of 95%

Error versus uncertainty

- It is important not to confuse the terms ‘**error**’ and ‘**uncertainty**’
  - **Error** is the difference between the measured value and the ‘true value’ of the thing being measured
  - **Uncertainty** is a quantification of the doubt about the measurement result
- Whenever possible we try to correct for any known errors
  - Corrections from calibration certificates
  - But any error whose value we do not know is a source of uncertainty

Why is uncertainty of measurement important?

- Wish to make good quality measurements and to understand the results
- Making the measurements as part of
  - **Calibration**
  - **Test**
  - **Meet the tolerance**
- Read and understand a calibration certificate
- Writing specification for a test or measurement

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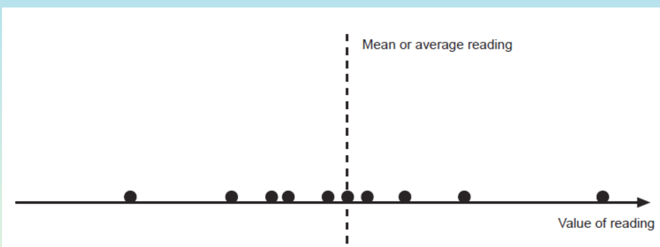
Basic statistical calculations

- When repeated measurements give different results...
- You can get some information from your measurements by carrying out some basic statistical calculations
  - Arithmetic mean: give us an estimate of the 'true' value

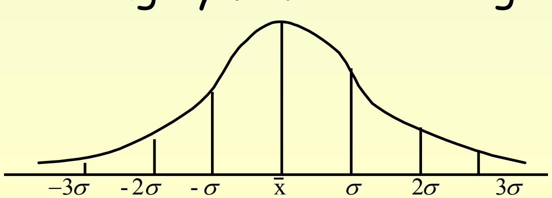


The readings are : 15,18,17,15,16,18,19,14,16 and 12

The sum of these is : 160

The average of the 10 reading is : 160/10 = 16

  - Standard deviation: know how widely spread the readings are and tell us something about the uncertainty of measurement
    - roughly 2/3 reading $\pm 1\sigma$
    - roughly 95.4% reading $\pm 2\sigma$



1. Difference between each reading and the average
   i.e. -1 +2 +1 -1 0 +2 +3 -2 0 -4
2. And square each of these
   i.e. 1 4 1 1 0 4 9 4 0 16
3. Find the total and divide by n-1
   i.e. (1+4+1+1+0+4+9+4+0+16)/9 = 4.44
4. The estimated standard deviation, s, by taking the square root of the total,

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{(n-1)}}$$

$$s = \sqrt{4.44} = 2.1$$

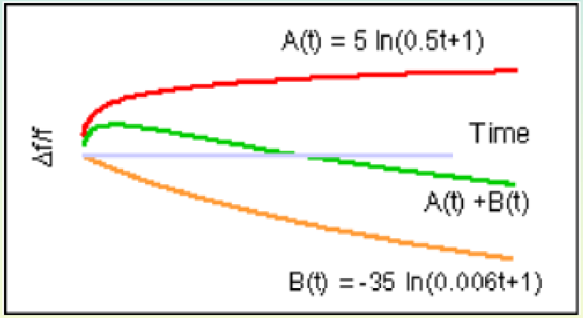
Statistical calculations in the field of time and frequency

- Standard deviation won't work for oscillators because it has aging effect. Therefore, the mean is not fixed and the deviation goes to infinity
- Instead, Allan Deviation $\sigma_y(\tau)$ and Time Deviation $\sigma_x(\tau)$ are adopted in the filed of time and frequency
- **Allan Deviation $\sigma_y(\tau)$ (Frequency)**

$$\sigma_y(\tau) = \sqrt{\frac{1}{2(M-1)}\sum_{i=1}^{M-1}(y_{i+1}-y_i)^2}$$

- **Time Deviation $\sigma_x(\tau)$ (Time)**

$$\sigma_x(\tau) = \frac{\tau}{\sqrt{3}} \operatorname{mod}\sigma_y(n\tau_0)$$



Aging Curve for Crystals
Aging effect can cause output frequency differ from the desired specified frequency

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Where do errors and uncertainties come from?

- If the size and effect of an error are known, a correction can be applied to the measurement result
- Many things can undermine a measurement (flaws)
  - Visible
  - Invisible
- Uncertainties from each of these sources, and from other sources, would be individual 'inputs' contributing to the overall uncertainty in the measurement
- Since real measurements are never made under perfect conditions, errors and uncertainties can come from :
  - The measuring instrument (bias、aging、drift、noise…)
  - The item being measured (not be stable)
  - The measurement process (difficult)
  - 'Imported' uncertainties (calibration)
  - Operator skill
  - Sampling issues
  - The environment (temperature、air pressure、humidity…)

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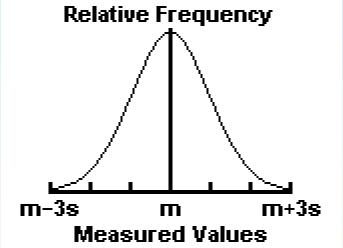
Random or systematic

- The effects that give rise to uncertainty in measurement can be either:

  - **Random** - where repeating the measurement gives a randomly different result

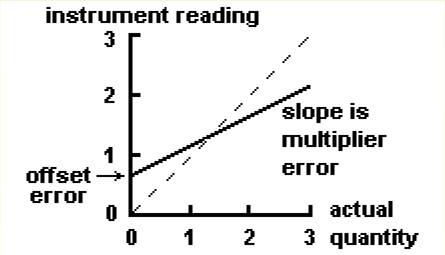
Precision



1. **Random errors often have a Gaussian normal distribution**
2. **The mean *m* of a number of measurements of the same quantity is the best estimate of that quantity**
3. **The standard deviation *s* of the measurements shows the accuracy of the estimate**

  - **Systematic** - where the same influence affects the result for each of the repeated measurements

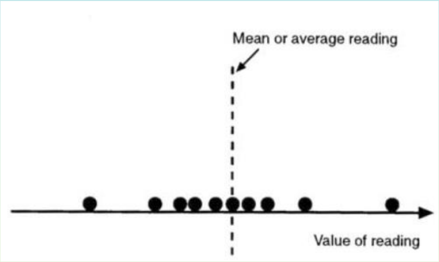
Accuracy

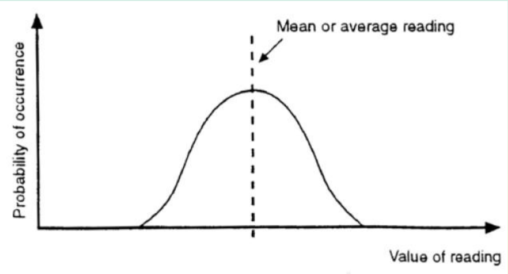


1. **Systematic errors in experimental observations usually come from the measuring instruments**

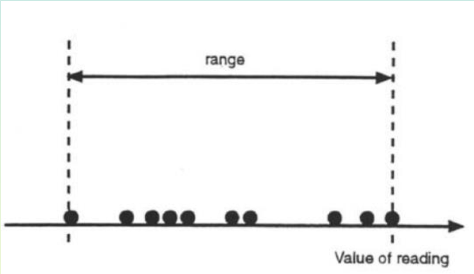
Distribution the 'shape' of the errors

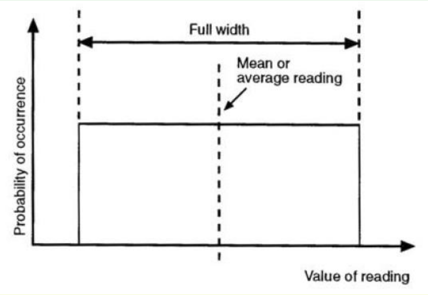
- Normal distribution
  - sometimes the values are more likely to fall near the average than further away





- Uniform or rectangular distribution
  - When the measurements are quite evenly spread between the highest and the lowest values





- Other distributions
  - distributions can have other shapes, for example, triangular , M-shaped, or lop-sided

What is not a measurement uncertainty?

- Mistakes made by operators
  - They should not be counted as contributing to uncertainty
  - They should be avoided by working carefully and by checking work
- Tolerances
  - They are acceptance limits which are chosen for a process or a product
- Specifications
  - A specification tells you what you can expect from a product

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How to calculate uncertainty of measurement

- To calculate the uncertainty of a measurement

1) Must identify the sources of uncertainty in the measurement
2) Estimate the size of the uncertainty from each source
3) Finally the individual uncertainties are combined to give an overall figure

The two ways to estimate uncertainties

- No matter what are the sources of your uncertainties, there are two approaches to estimating them
  - **Type A evaluations** - Uncertainty estimates using statistics (usually from repeated readings)
  - **Type B evaluations** - Uncertainty estimates from any other information
    - past experience of the measurements
    - calibration certificates
    - manufacturer's specifications
    - calculations
    - published information
    - common sense

Eight main steps to evaluating uncertainty

① Decide what you need to find out from your measurements. Decide what actual measurements and calculations are needed to produce the final result

② Carry out the measurements needed

③ Estimate the uncertainty of each input quantity that feeds into the final result. Express all uncertainties in similar terms

④ Decide whether the errors of the input quantities are independent of each other. If you think not, then some extra calculations or information are needed

⑤ Calculate the result of your measurement (including any known corrections for things such as calibration)

⑥ Find the combined standard uncertainty from all the individual aspects

⑦ Express the uncertainty in terms of a coverage factor, together with a size of the uncertainty interval, and state a level of confidence

⑧ Write down the measurement result and the uncertainty, and state how you got both of these

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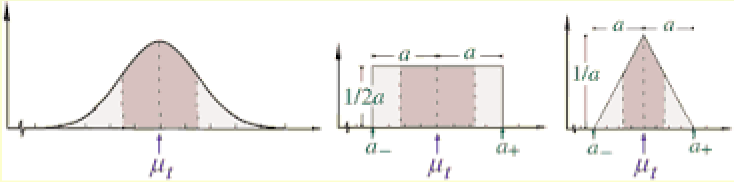
- Uncertainty contributions must be expressed in similar terms before they are combined
  - given in the same units, at the same level of confidence
  - converting them into standard uncertainties
- **Standard uncertainty** u, or u(y)
  - Tells us about the uncertainty of an average (not just about the spread of values)
  - **Type A evaluation** (several repeated reading has been taken)
    - Estimated standard uncertainty $u = \frac{s}{\sqrt{n}}$ *estimated standard deviation, s* *the number of measurements, n*
    - Allan Deviation $\sigma_y(\tau)$ (Frequency) $\sigma_y(\tau) = \sqrt{\frac{1}{2(M-1)} \sum_{i=1}^{M-1} (y_{i+1} - y_i)^2}$
    - Time Deviation $\sigma_x(\tau)$ (Time) $\sigma_x(\tau) = \frac{\tau}{\sqrt{3}} \mathrm{mod}\,\sigma_y(n\tau_0)$
  - **Type B evaluation** (where the information is more scarce)
    - So we need to estimate the upper and lower limits of uncertainty
      - Normal $u(\mathrm{x}) = \frac{a}{\sqrt{1}}$
      - Rectangular $u(\mathrm{x}) = \frac{a}{\sqrt{3}}$
      - Triangular $u(\mathrm{x}) = \frac{a}{\sqrt{6}}$



- Uncertainty contributions must be in the same units before they are combined
  - Ex: in making a measurement of length, the measurement uncertainty will also eventually be stated in terms of length
    - Uncertainty source – temperature
    - Effect – length 0.1%/ ℃
    - If a temperature uncertainty of ±2 ℃ --> ±0.2 cm in a piece of the material 100 cm long
- Individual standard uncertainties calculated by Type A or Type B evaluations can be combined validly by root sum of the squares (**the combined standard uncertainty, u_c**)
  - By squaring the uncertainties, adding them all together
  - Taking the square root of total

$$Combined\ uncertainty = \sqrt{a^2 + b^2 + c^2 + ..etc.}$$

- Correlation
  - The equation given above is correct if all the uncertainty contributions are independent
  - But if they are not, extra calculations are needed
- Coverage factor k
  - Re-scaling u_c by a coverage factor, k we will have an overall uncertainty (expanded uncertainty, $U = ku_c$) stated at another level of confidence

coverage factors k for a normal distribution

k = 1, approximately 68 % confidence level

k = 2, approximately **95 %** confidence level

k = 3, approximately 99.7% confidence level

How to express the answer

- It is important to express the answer so that a reader can use the information
  - The measurement result, together with the uncertainty figure
    - e.g. The length of the stick was 20 cm ±1 cm
  - The statement of the coverage factor and the level of confidence
    - The reported uncertainty is based on a standard uncertainty multiplied by a coverage factor $k = 2$, providing a level of confidence of approximately 95%
  - How the uncertainty was estimated

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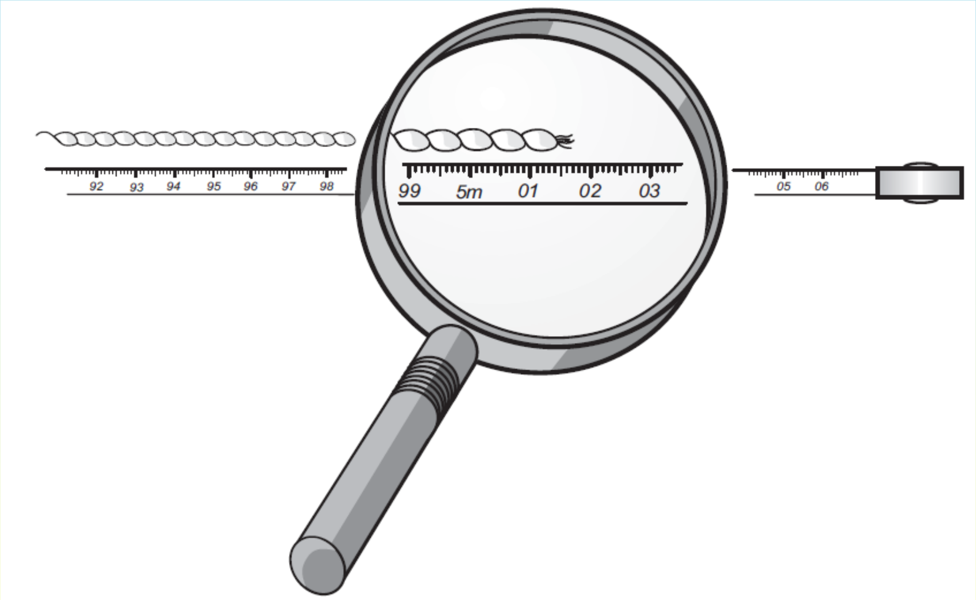
Example1 for a simple uncertainty analysis

- Following [the steps mentioned above]
- How long is a piece of string?



- **Step 1.** Decide what you need to find out from your measurements. Decide what actual measurements and calculations are needed to produce the final result
  - You need to make a measurement of the length, using a tape measure. Apart from the actual length reading on the tape measure, you may need to consider
  - ➢ Possible errors of the tape measure
    - Does it need any correction, and what is the uncertainty in the calibration?
    - Could bending have shortened it?
    - What is the resolution, i.e. how small are the divisions on the tape?
  - ➢ Possible errors due to the item being measured
    - Does the string lie straight?
    - Does the prevailing temperature or humidity (or anything else) affect its actual length?
  - ➢ Possible errors due to the measuring process, and the person making the measurement
    - How well can you line up the beginning of the string with the beginning of the tape measure?
    - Can the tape be laid properly parallel with the string?
    - How repeatable is the measurement?

- **Step 2.** Carry out the measurements needed
  - Make and record your measurements of length
  - To be extra thorough, you need repeat the measurement
  - Let us suppose you calculate the mean to be 5.017 m, and the estimated standard deviation to be 0.0021 m (i.e. 2.1 millimetres)

  For a careful measurement you might also record:

  ✓ when you did it, how you did it

  ✓ environmental conditions and anything else that could be relevant
- **Step 3.** Estimate the uncertainty of each input quantity that feeds into the final result. Express all uncertainties in similar terms (standard uncertainty, u)
  - look at all the possible sources of uncertainty
  - estimate the magnitude of each

  Type B evaluation

  ✓ The tape measure has been calibrated. It needs no correction, but the calibration uncertainty is 0.1 percent of reading, at a coverage factor k = 2 (for a normal distribution). In this case, 0.1 percent of 5.017 m is close to 5 mm. Dividing by 2 gives the standard uncertainty (for k = 1) to be u = 2.5 mm

- ✓ The divisions on the tape are millimetres. Reading to the nearest division gives an error of no more than ±0.5 mm. We can take this to be a uniformly distributed uncertainty. To find the standard uncertainty, u, we divide the half-width (0.5 mm) by √3, giving u = 0.3 mm, approximately
- ✓ The tape lies straight, but let us suppose the string unavoidably has a few slight bends in it. Therefore the measurement is likely to underestimate the actual length of the string. Let us guess that the underestimate is about 0.2 percent, and that the uncertainty in this is also 0.2 percent at most. That means we should correct the result by adding 0.2 percent (i.e. 10 mm). The uncertainty is assumed to be uniformly distributed, in the absence of better information. Dividing the half-width of the uncertainty (10 mm) by √3 gives the standard uncertainty u = 5.8 mm

Type A evaluation

- ✓ The standard deviation tells us about how repeatable the placement of the tape measure is, and how much this contributes to the uncertainty of the mean value. The estimated standard deviation of the mean of the 10 readings is found

$$u = \frac{s}{\sqrt{n}} = \frac{2.1}{\sqrt{10}} = 0.7mm$$

- **Step 4.** Decide whether the errors of the input quantities are independent of each other
  - In this case, let us say that they are all independent
- **Step 5.** Calculate the result of your measurement (including any known corrections for things such as calibration)
  - The result comes from the mean reading, together with the correction needed for the string lying slightly crookedly

i.e. 5.017 m + 0.010 m = 5.027 m

- **Step 6**. Find the combined standard uncertainty from all the individual aspects
  - standard uncertainties are combined as

$$Combined\ \text{standard uncertainty} = \sqrt{2.5^2 + 0.3^2 + 5.8^2 + 0.7^2} = 6.4mm$$

- **Step 7**. Express the uncertainty in terms of a coverage factor, together with a size of the uncertainty interval, and state a level of confidence
  - For a coverage factor k = 2 to give an expanded uncertainty of 12.8 mm (i.e. 0.0128 m). This gives a level of confidence of about 95 percent

- **Step 8**. Write down the measurement result and the uncertainty, and state how you got both of these. You might record
  - The length of the string was 5.027 m ±0.013 m. The reported expanded uncertainty is based on a standard uncertainty multiplied by a coverage factor k = 2, providing a level of confidence of approximately 95%
  - The reported length is the mean of 10 repeated measurements of the string laid horizontally. The result is corrected for the estimated effect of the string not lying completely straight when measured

Uncertainty budget

- To help in the process of calculation, it can be useful to summarize the uncertainty analysis or uncertainty budget in a spreadsheet as following

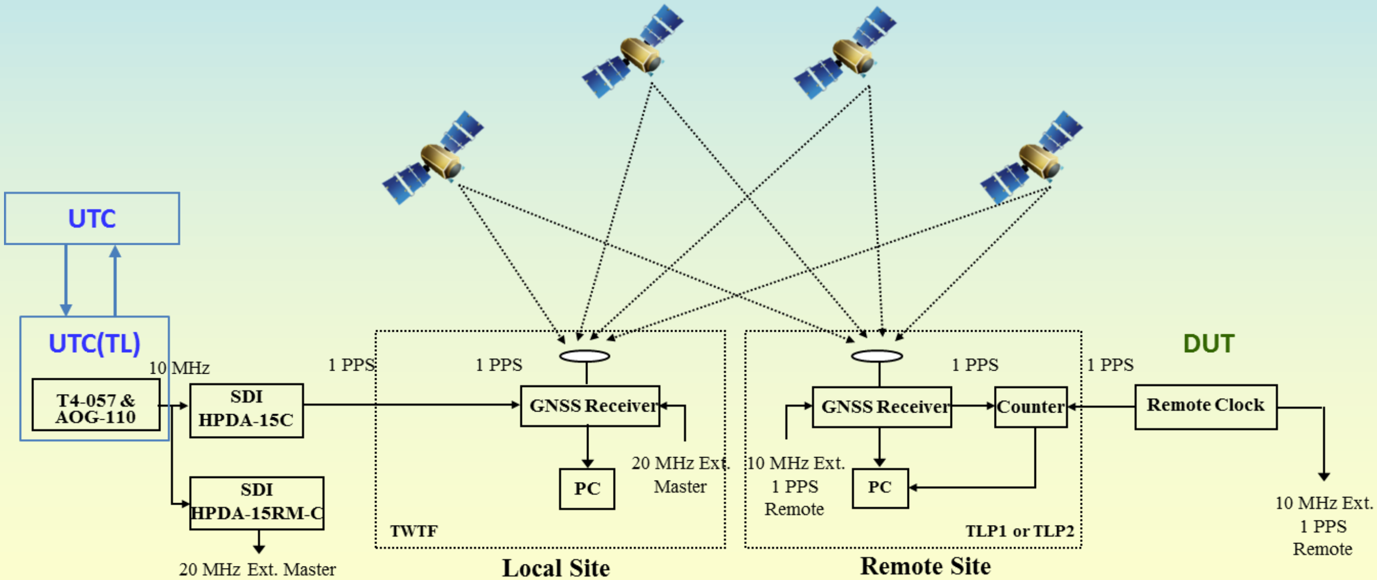
Source of uncertainty	Value ±	Probability distribution	Divisor	Standard uncertainty
Calibration uncertainty	5.0 mm	Normal	2	2.5 mm
Resolution(size of divisions)	0.5 mm	Rectangular	√3	0.3 mm
String not lying perfectly straight	10.0 mm	Rectangular	√3	5.8 mm
Standard uncertainty of mean of 10 repeated readings	0.7 mm	Normal	1	0.7 mm
Combined standard uncertainty		Assumed normal		6.4 mm
Expanded uncertainty (a level of confidence 95%)		Assumed Normal (k=2)		12.8 mm

Reference

1. Stephanie Bell, *A Beginner's Guide to Uncertainty of Measurement, 2001.*
2. *Guide to the Expression of Uncertainty in Measurement*，ISO，1995
3. Jeff Cartright, *Aging Performance in Crystals*, 2008
4. BIPM, IEC, IFCC, ISO, IUPAC, IUPAP, OIML. *Guide to the Expression of Uncertainty in Measurement.* International Organization for Standardization, Geneva. ISBN 92-67-10188-9, First Edition 1993, corrected and reprinted 1995. (BSI Equivalent: BSI PD 6461: 1995, Vocabulary of Metrology, Part 3. Guide to the Expression of Uncertainty in Measurement. British Standards Institution, London.)
5. https://www.google.com.tw/search?q=uncertainty+comic&hl=zh-TW&rlz=1T4WQIB_zh-TWTW565TW566&tbm=isch&tbo=u&source=univ&sa=X&ved=0ahUKEwikpqaQ_p_PAhVIUZQKHVewB7YQsAQIGw&biw=1920&bih=841#imgdii=RwHdyPlz56-FFM%3A%3BRwHdyPlz56-FFM%3A%3BFrblpQIAImXSdM%3A&imgrc=RwHdyPlz56-FFM%3A
6. http://www.physics.umd.edu/courses/Phys276/Hill/Information/Notes/ErrorAnalysis.html

Example2 for GPS remote time calibration system uncertainty analysis

- Following the steps mentioned above
- Uncertainty Budget of Time and Frequency Calibration System at TL_.pdf



Uncertainty budget

Type	Uncertainty component	Uncertainty (ns)	Coverage factor	Probability density distribution	Measurement Uncertainty (ns)
B	Receiver resolution at TL	4.0	$\sqrt{3}$	rectangular	2.31
B	Receiver resolution at Remote	10.0	$\sqrt{3}$	rectangular	5.77
B	Ephemeris error	12.0	$\sqrt{3}$	rectangular	6.93
B	Common-View Common-Clock peak-to-peak variation	5.0	$\sqrt{3}$	rectangular	2.89
B	Antenna coordinate error	6.0	$\sqrt{3}$	rectangular	3.46
B	Environmental factor	3.0	$\sqrt{3}$	rectangular	1.73
B	Multipath	5.9	$\sqrt{3}$	rectangular	3.41
B	Delay error of Ionosphere	2.0	$\sqrt{3}$	rectangular	1.15
B	Delay error of Troposphere	3.0	$\sqrt{3}$	rectangular	1.73
B	Traceability of UTC(TL) to UTC from Circular T	15.7	$\sqrt{3}$	rectangular	9.06
B	Uncertainty of UTC(TL) v.s. UTC	5.2	1	normal	5.2
B	Time interval counter	1.0	$\sqrt{3}$	rectangular	0.58
B	RG 58 cable temp. influence	0.08	$\sqrt{3}$	rectangular	0.046
B	LMR-400 cable temp. influence	0.022	$\sqrt{3}$	rectangular	0.013
B	Jitter of 1 PPS	0.1	1	normal	0.10
B	Unidentified variation	5.0	$\sqrt{3}$	rectangular	2.89
A	Common-View result between two sites UTC(TL) - DUT	2.0	1	normal	2.00
Expanded time uncertainty with a coverage factor of k=1					15.74

Thank you